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ERROR ESTIMATION IN FINITE ELASTICITY AND PLASTICITY PROBLEMS

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Abstract. *This paper describes a methodology for error estimation with enhanced assumed strain elements applied to linear solid mechanics problems, finite elasticity problems and plasticity.*
The relation between the enhanced strain modes and the quality of the finite element solution is analysed for problems of solid mechanics. The analysis is developed in the context of error estimation. The contribution of the enhanced strain modes is quantified with an energy norm. The methodology proposed for error estimation has the advantages of a) being a local formulation, b) computing the error in an element-by-element way, and c) having a simple interpretation from a practical point of view.
In the paper firstly the general formulation of the error estimator is described. Following it is applied to linear and non-linear elasticity and plasticity problems. Representative numerical simulations are presented for 3D non linear elasticity and Von Mises plasticity, with emphasis in the distribution of the local error and the global rate of convergence.

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1 INTRODUCTION

The finite element method is a computational tool widely used in the design and verification of engineering structures. However, obtaining a value of the finite element solution is not the only issue, it is also necessary to assess the quality of the computed results.

The interest of "a posteriori" error estimators lies on their direct applicability to adaptive refinement techniques. The development of these techniques began in the seventies with the pioneering works of Babuška et al [1]. From this time till now, several error estimators have been proposed for linear analyses, whose efficiency has been proved in a wide variety of problems.

Nevertheless, developments in non linear problems have not been made until recently, and a number of research lines remain open. We remark the developments of Ortiz and Quigley [2] in localisation, Johnson and Hansbo [3] in the context of the elastic-plastic model of Hencky, the error estimator of Barthold et al [4] that is applied to the elastic-plastic models of Hencky and Prandtl-Reuss, etc. Finally it's necessary to point out the recent works of Radovitzky and Ortiz [5] in error estimation for highly non linear problems, including finite deformations in hyperelasticity, viscoplasticity, dynamics, etc.

In this paper a methodology for error estimation in linear and non linear problems is described. The proposed method gives a bound of the discretisation error associated to the finite element solution computed with the standard displacement formulation. This error is computed through the enhanced assumed strain [6, 7] finite element solution (section 2). For error estimation a variational structure of the boundary value problem is required. This requirement and its influence in local and global error estimation is analysed in section 3. The general expression of the error estimator is described in section 4. The details corresponding to finite elasticity problems and Von Mises plasticity are explained in section 5. Finally, in section 6 some representative numerical simulations are shown, and section 7 describes the conclusions.

2 ENHANCED ASSUMED STRAIN FINITE ELEMENT FORMULATION

The *enhanced assumed strain* (EAS) finite element formulation [6, 7, 8, 9] is based on the discrete variational equations obtained from the Hu-Washizu functional [10].

The existence of a function of internal energy per unit of volume, in each point $\boldsymbol{x} \in \Omega$, is assumed. This function may be expressed as function of the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ for linear problems, or the deformation gradient $\boldsymbol{F}$ for finite deformation problems.

For infinitesimal strains the key ingredient is the additive decomposition of the strain field in a compatible part and an enhanced part:

$$\boldsymbol{\varepsilon} = \underbrace{\nabla^s \boldsymbol{u}}_{\text{compatible}} + \underbrace{\tilde{\varepsilon}}_{\text{enhanced}} ; \tag{1}$$

where $\nabla^s \boldsymbol{u}$ (symmetric component of displacement gradient) is the *"compatible"* part of strain field, and $\tilde{\varepsilon}$ is the enhanced (or incompatible) one. This denomination is motivated for the enhancement of the approximated solution associated with the incompatible part in discrete meshes (for the exact solution the field $\tilde{\varepsilon}$ is null). There are no requirements of inter-element continuity for the enhanced field $\tilde{\varepsilon}$.

In EAS formulation for finite deformation problems, the deformation gradient is parametrised via the following additive decomposition:

$$\boldsymbol{F} = \underbrace{\boldsymbol{\nabla}_X \boldsymbol{\varphi}}_{\text{compatible}} + \underbrace{\tilde{\boldsymbol{F}}}_{\text{enhanced}} \tag{2}$$

where $\boldsymbol{\nabla}_X$ is the gradient operator and $\boldsymbol{\varphi}$ is the deformation mapping.

3 ERROR ESTIMATION METHODOLOGY BASED ON ENERGY NORMS

This section describes the general framework for the error estimation methodology. To this end the variational structure of the boundary value problem, the methodology of approximation via finite element method and the requirements for the formulation of local and global error estimators, are explained.

3.1 Variational structure of the boundary value problem

Consider the classical boundary value problem for the equilibrium of a solid $\Omega \cup \partial\Omega$:

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0} \qquad & \text{in } \Omega \cup \partial\Omega \\ \boldsymbol{u} = \overline{\boldsymbol{u}} \qquad & \text{in } \partial_u\Omega \\ \boldsymbol{\sigma}\boldsymbol{n} = \overline{\boldsymbol{t}} \qquad & \text{in } \partial_t\Omega \end{aligned} \tag{3}$$

being the displacements $\boldsymbol{u}$ the unknown field, $\boldsymbol{\sigma}$ the Cauchy stress tensor, $\boldsymbol{b}$ the body forces, $\boldsymbol{n}$ the normal vector in $\partial\Omega$, and $\overline{\boldsymbol{t}}$, $\overline{\boldsymbol{u}}$ prescribed values. If the boundary value problem (3) has variational structure, the Dirichlet form $a(\boldsymbol{u})[\boldsymbol{\eta}, \boldsymbol{\eta}]$ associated to the functional $\Pi(\boldsymbol{u})$ is defined as:

$$a(\boldsymbol{u})[\boldsymbol{\eta}, \boldsymbol{\eta}] = \int_\Omega \frac{\partial^2 W(\boldsymbol{\varepsilon}, \boldsymbol{x})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} \eta_{i,j} \eta_{k,l} d\Omega \tag{4}$$

being $\boldsymbol{\eta} \in V : \Omega \to \mathbb{R}^n$ admissible variations, V is the space of functions with finite energy and W is the function of internal energy density.

The Dirichlet form $a(\boldsymbol{u})[\boldsymbol{\eta}, \boldsymbol{\eta}]$ is *regular* if the following conditions are verified:

1. $$C_{ijkl} = \frac{\partial^2 W(\boldsymbol{\varepsilon}, \boldsymbol{x})}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} < \infty, \; \forall \boldsymbol{x} \in \Omega \Leftrightarrow C_{ijkl} \in L^\infty(\Omega, \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \tag{5}$$
2. $$a(\boldsymbol{u})[\boldsymbol{\eta}, \boldsymbol{\eta}] > C \|\boldsymbol{\eta}\|^2_{1,2} \quad C \in \mathbb{R}^+ \text{ (coercivity condition)} \tag{6}$$

where L^{∞} is the Lebesgue space of order infinity and $\|\cdot\|_{1,2}$ is the Sobolev norm with degree 1 and order 2.

If the Dirichlet form (4) verifies the regularity hypotheses (5, 6), then the following conditions are asserted:

i) Π is convex

ii) Π has a unique relative minimum; hence, the solution $\boldsymbol{u}$ of the boundary value problem verifies:

$$\Pi(\boldsymbol{u}) = \inf_{\boldsymbol{v}\in V} \Pi(\boldsymbol{v}) \tag{7}$$

3.2 Methodology of approximation with compatible elements

For infinitesimal elasticity, the variational equation of the principle associated to the functional $\Pi(\boldsymbol{u})$ is:

$$G(\boldsymbol{u})[\boldsymbol{\eta}] = 0 \qquad \forall \boldsymbol{\eta} \in V \tag{8}$$

being $G(\boldsymbol{u})[\boldsymbol{\eta}]$ the weak form derived from the boundary value problem (3):

$$G(\boldsymbol{u})[\boldsymbol{\eta}] \stackrel{\text{def}}{=} \int_{\Omega} \operatorname{div} \boldsymbol{\sigma} \cdot \boldsymbol{\eta}\, d\Omega - \int_{\Omega} \boldsymbol{b} \cdot \boldsymbol{\eta}\, d\Omega - \int_{\partial_t \Omega} \bar{\boldsymbol{t}} \cdot \boldsymbol{\eta}\, d\Gamma \tag{9}$$

Let $V_h \subset V$ be a finite dimension subspace of V, such that V_h approaches V when $h \to 0$. If the restriction of (8) only to variations $\boldsymbol{\eta}_h \in V_h$:

$$G(\boldsymbol{u})[\boldsymbol{\eta}_h] = 0 \qquad \forall \boldsymbol{\eta}_h \in V_h \tag{10}$$

is subtracted from the particularisation of (8) to elements of V_h (displacements and variations):

$$G(\boldsymbol{u}_h)[\boldsymbol{\eta}_h] = 0 \qquad \forall \boldsymbol{\eta}_h \in V_h \tag{11}$$

the following result is obtained if the weak form $G(\boldsymbol{u})[\boldsymbol{\eta}]$ is linear in $\boldsymbol{u}$:

$$a(\boldsymbol{u})[\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{\eta}_h] = 0 \quad \forall \boldsymbol{\eta}_h \in V_h \tag{12}$$

Equation (12) establishes that the finite element solution minimises the value of $\|\boldsymbol{u} - \boldsymbol{u}_h\|_E$. This property is referred to as the *optimal approximation property* of the finite element method:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_E = \inf_{\boldsymbol{v_h}\in V_h} \|\boldsymbol{u} - \boldsymbol{v}_h\|_E \tag{13}$$

3.3 Local error estimation

In general, the finite element solution is obtained in the discrete domain Ω^h, which is constructed via the discretisation of the domain Ω, using n_{el} elements Ω^e, such that:

$$\begin{aligned} \bigcup_{e=1}^{n_{el}} \Omega^e &= \Omega_h \\ \Omega_i^e \cap \Omega_j^e &= \emptyset \quad \forall i \neq j \end{aligned}$$

Let Ω^e be an element in $\mathbb{R}^n$ with positive jacobian determinant, and let $\mathbb{P}_p(\Omega^e)$ be the set of polynomials over Ω^e with degree lower or equal than p. Let $\boldsymbol{u}^e \in H^1(\Omega^e, \mathbb{R}^n)$ be the exact displacement field in the element e, and let $\boldsymbol{u}_h^e(\boldsymbol{x}) = \sum_{a=1}^{n_{node}} \boldsymbol{u}_a N_a(\boldsymbol{x}) \in \mathbb{P}_p(\Omega^e)$ be the "finite dimension interpolant polynomial" of the exact solution $\boldsymbol{u}^e$, where n_{node} is the number of nodes of the element e.

The local error function in the element e is defined as the difference between the exact displacement field and the displacement field computed via the finite element method:

$$\boldsymbol{E}^e(\boldsymbol{x}) = \boldsymbol{u}^e(\boldsymbol{x}) - \boldsymbol{u}_h^e(\boldsymbol{x})$$

The problem to solve with a local error estimator is to obtain an upper bound of the local error function, which may be expressed in the following way:

$$\|\boldsymbol{u}^e - \boldsymbol{u}_h^e\| \leq C(h^e)^\alpha |\boldsymbol{u}^e| \tag{14}$$

where:

C	:	real positive constant
h^e	:	diameter of the circunference circumscribed around Ω^e
$\|\boldsymbol{u}^e\|$	:	seminorm of $\boldsymbol{u}^e$
α	:	rate of convergence

The definition of the semi-norm used in (14) is independent of the definition of the error norm established. The equality (14) is verified if the *optimal approximation property* (13) and the regularity conditions expressed in (5; 6) are satisfied.

3.4 Global error estimation

From the expression of the interpolation functions of $\boldsymbol{u}^e(\boldsymbol{x})$,

$$\boldsymbol{u}_h^e(\boldsymbol{x}) = \sum_{a=1}^{n_{node}} \boldsymbol{u}_a N_a(\boldsymbol{x})$$

the *Global interpolant polynomial* $\boldsymbol{u}_h(\boldsymbol{x})$ is defined as:

$$\boldsymbol{u}_h(\boldsymbol{x}) = \sum_{e=1}^{n_{el}} \boldsymbol{u}_h^e(\boldsymbol{x})$$

If the shape functions are conforming, the following is satisfied: $\boldsymbol{u}_h^e(\boldsymbol{x}) \in H^1(\Omega^e, \mathbb{R}^n) \Rightarrow \boldsymbol{u}_h(\boldsymbol{x}) \in H^1(\Omega, \mathbb{R}^n)$, where H^1 is the Sobolev space of order 1.

In order to write an upper bound of the global error function: $\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{u}_h(\boldsymbol{x})$, the seminorm of $\boldsymbol{E}(\boldsymbol{x})$ used in (14) is expressed as the summation of the contributions of each element. Using the energy norm, this results in [11]:

$$|\boldsymbol{u} - \boldsymbol{u}_h|_{1,2} \leq \sum_{e=1}^{n_{el}} C \frac{(h^e)^2}{\rho^e} |\boldsymbol{u}^e|_{2,2} \tag{15}$$

The expression (15) shows that the upper bound of the global error may be expressed as the sum of the local error bounds computed in each element. Besides, if regularity conditions (5,6) hold and taking into account the inequality of Poincaré, the semi norm $|\cdot|_{1,2}$ can be replaced by the energy norm in (15), resulting:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_E \leq C \sum_{e=1}^{n_{el}} \frac{(h^e)^2}{\rho^e} |\boldsymbol{u}^e|_{2,2} \tag{16}$$

4 ERROR ESTIMATOR PROPOSED

From a practical point of view, equation (16) is not convenient because the error is expressed in terms of the unknown exact solution $\boldsymbol{u}^e$. Besides it is not possible to substitute this field by its approximate solution $\boldsymbol{u}_h^e$, as it is a polynomial of degree k and the seminorm used is of order $k+1$ ($D^{k+1}\boldsymbol{u}_h^e = 0$).

Error estimation techniques are based on the substitution of $\boldsymbol{u}^e$ by another field, in such manner that the estimated error must be a realistic measure. The methodology for performing this substitution leads to different error estimators.

The error estimator for the solution $\boldsymbol{u}_h$ (obtained with elements formulated in displacements) analysed in this paper is based on the solution $\boldsymbol{u}_{\text{enh}}$ obtained with the enhanced assumed strain elements described in section 2.

The starting point is the triangular inequality:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_E \leq \|\boldsymbol{u} - \boldsymbol{u}_{\text{enh}}\|_E + \|\boldsymbol{u}_{\text{enh}} - \boldsymbol{u}_h\|_E \tag{17}$$

It is assumed that the rates of convergence are:

$$\|\boldsymbol{u} - \boldsymbol{u}_{\text{enh}}\|_E = o(h^m) \tag{18}$$

$$\|\boldsymbol{u}_{\text{enh}} - \boldsymbol{u}_h\|_E = o(h^p) \tag{19}$$

Also, at least in the asymptotic regime, the following hypothesis holds:

$$m > p \tag{20}$$

In these conditions, at least for $h \to 0$, in the right hand side of equation (17) the first term is negligible if it is compared to the second one. Therefore it is possible to establish that:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_E \leq C\|\boldsymbol{u}_{\text{enh}} - \boldsymbol{u}_h\|_E \quad C \in \mathbb{R}^+ \tag{21}$$

The hypotheses (18;19;20) may be re-interpreted in the following terms: The solutions $\boldsymbol{u}_{\text{enh}}$ and $\boldsymbol{u}_h$ converge to the exact solution in such manner that

1. $\|\boldsymbol{u}_{\text{enh}} - \boldsymbol{u}_h\|_E$ decreases with the refinement of the mesh;
2. The solution obtained with enhanced elements is a better approximation to the exact solution than the solution of standard elements to the enhanced ones.

The expression of the local estimator proposed is:

$$(E^e)^2 = \|\boldsymbol{u}^e_{\text{enh}} - \boldsymbol{u}^e_h\|_E \tag{22}$$

In accordance to the previous section, the global error may be obtained as the sum of the local errors:

$$E^2 = \sum_{i=1}^{n_{el}} (E^i)^2 \tag{23}$$

The discretisation error associated to the standard elements is quantified via the internal energy associated to the incompatible modes computed with enhanced elements.

Each component in the sum (23) is local, and therefore the proposed estimator has the important advantage that is computed element by element, without global smoothing techniques nor subdomain solutions.

5 ENERGY CONTRIBUTION OF THE INCOMPATIBLE MODES

In this section the application of (22) to error estimation in non-linear problems is explained. Finite elasticity problems with hyperelastic constitutive models and small strain problems with Von Mises plasticity are considered.

5.1 Finite elasticity

Here the unknown field is the deformation mapping $\boldsymbol{\varphi} : \Omega \rightarrow \Omega_t$, where Ω is the reference configuration and Ω_t is the deformed configuration at time t. The formulation is similar to what has been already developed in section 3.1, but replacing the displacement field $\boldsymbol{u}$ for the deformation $\boldsymbol{\varphi}$, and the infinitesimal strain tensor $\boldsymbol{\varepsilon}$ for the deformation gradient $\boldsymbol{F}$.

With respect to the approximation methodology via standard elements described in 3.2, subtracting (10; 11) the following result is obtained:

$$G(\boldsymbol{\varphi})[\boldsymbol{\eta}_h] - G(\boldsymbol{\varphi}_h)[\boldsymbol{\eta}_h] = 0 \quad \forall \boldsymbol{\eta}_h \in V_h \tag{24}$$

This is different to (12), as the Dirichlet form $a(\boldsymbol{\varphi})[\cdot,\cdot]$ is *non linear* in finite elasticity. Nevertheless, for the asymptotic regime $(h \rightarrow 0)$, the finite element solution $\boldsymbol{\varphi}_h$ is approximately equal to the exact solution, and then equation (24) may be linearised resulting in:

$$a(\boldsymbol{\varphi})[\boldsymbol{\varphi} - \boldsymbol{\varphi}_h, \boldsymbol{\eta}_h] = 0 \quad \forall \boldsymbol{\eta}_h \in V_h, \quad h \rightarrow 0 \tag{25}$$

This condition establishes the *optimal approximation property* of the finite element method, for finite elasticity, in the asymptotic regime:

$$\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_E = \inf_{\boldsymbol{v_h} \in V_h} \|\boldsymbol{\varphi} - \boldsymbol{v}_h\|_E \tag{26}$$

The expression of the local error estimator proposed in (22) results:

$$(E^e)^2 = \|\boldsymbol{\varphi}^e_{\text{enh}} - \boldsymbol{\varphi}^e_h\|_E \tag{27}$$

assuming the hypotheses (18; 19; 20) hold.

For the numerical implementation, the value of (27) is computed in the reference configuration. Then, the expression of the energy norm is [12]:

$$\|\boldsymbol{\varphi}\|^2_E = a(\boldsymbol{\varphi})[\boldsymbol{\varphi}, \boldsymbol{\varphi}] = \int_{\Omega_0} \boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{\varphi} \cdot \mathbf{A} \boldsymbol{\nabla}_{\boldsymbol{X}} \boldsymbol{\varphi} \, d\Omega \tag{28}$$

where $\mathbf{A}$ is the tangent tensor of constitutive moduli:

$$\mathbf{A} = \frac{\partial^2 W(\boldsymbol{X}, \boldsymbol{F})}{\partial \boldsymbol{F} \partial \boldsymbol{F}} = \frac{\partial \boldsymbol{P}}{\partial \boldsymbol{F}} \tag{29}$$

Simple calculations provide the expression of the error estimator that has been implemented [12]:

$$(E^e)^2 = \frac{1}{2} \int_{\Omega^e} \widetilde{\boldsymbol{F}} \cdot \mathbf{A} \widetilde{\boldsymbol{F}} \, d\Omega \tag{30}$$

where $\widetilde{\boldsymbol{F}}$ is the enhanced part of the deformation gradient [9].

Computing the global error via (30) extended over all the domain Ω, it can be expressed as the sum of the local errors:

$$E^2 = \sum_{i=1}^{n_{el}} (E^i)^2 \tag{31}$$

5.2 Plasticity

The methodology for error estimation described in section 3 assumes a variational structure of the boundary value problem. In plasticity, this variational structure may be obtained at an incremental level via the variational integration of the plasticity equations [13]. The variational integration postulates the existence of an incremental energy function per unit volume $W_{t+\Delta t}$, such that

$$\boldsymbol{\sigma}_{t+\Delta t} = \frac{\partial W_{t+\Delta t}}{\partial \boldsymbol{\varepsilon}^e_{t+\Delta t}} \tag{32}$$

In infinitesimal J_2 plasticity with isotropic hardening, the functional dependence of $W_{t+\Delta t}$ is on elastic strain and effective plastic strain ξ. The expression of the incremental potential function is:

$$W_{t+\Delta t}(\boldsymbol{\varepsilon}^e_{t+\Delta t}, \xi_{t+\Delta t}, \boldsymbol{\varepsilon}^e_t, \xi_t) = \min_{\xi_{t+\Delta t}} \left(\Psi_{t+\Delta t}(\boldsymbol{\varepsilon}^e_{t+\Delta t}, \xi_{t+\Delta t}) - \Psi_t(\boldsymbol{\varepsilon}^e_t, \xi_t) \right) \tag{33}$$

where $\Psi(\boldsymbol{\varepsilon}^e, \xi)$ is the free energy function. The minimum requirement in the right-hand side of (33) is equivalent to the condition:

$$\frac{\partial \Psi_{t+\Delta t}(\boldsymbol{\varepsilon}^e_{t+\Delta t}, \xi_{t+\Delta t})}{\partial \xi_{t+\Delta t}} = 0 \tag{34}$$

Assuming that the elastic response is independent of the phenomena associated to unrecoverable distortions of the crystalline lattice, the free energy function may be expressed via the additive decomposition in an elastic part and a plastic part. Besides, if the additive decomposition of the infinitesimal strain tensor is assumed:

$$\Psi(\boldsymbol{\varepsilon}^e, \xi) = \Psi^e(\boldsymbol{\varepsilon}^e) + \Psi^p(\xi); \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p, \tag{35}$$

the incremental potential $W_{t+\Delta t}$ can be written as:

$$W_{t+\Delta t} = \min_{\xi_{t+\Delta t}} \left(\Psi^e_{t+\Delta t}(\boldsymbol{\varepsilon}_{t+\Delta t} - \boldsymbol{\varepsilon}^p_{t+\Delta t}) + \Psi^p_{t+\Delta t}(\xi_{t+\Delta t}) - \Psi^e_t(\boldsymbol{\varepsilon}_t - \boldsymbol{\varepsilon}^p_t) + \Psi^p_t(\xi_t) \right) \tag{36}$$

The optimisation condition (34) applied to (36), leads to the following expression [12]:

$$\left(J_{2,_{t+\Delta t}}\right)^2 = \frac{2}{3} \frac{\partial \Psi^p}{\partial \xi_{t+\Delta t}} \tag{37}$$

where J_2 is the second invariant of the deviatoric part of the stress tensor.

In this situation the Dirichlet form of the boundary value problem is:

$$a(\boldsymbol{u}_{t+\Delta t})[\boldsymbol{\eta}, \boldsymbol{\eta}] = \int_\Omega \boldsymbol{\nabla}^s \boldsymbol{\eta} \cdot \frac{\partial^2 W_{t+\Delta t}}{\partial \boldsymbol{\varepsilon}_{t+\Delta t} \partial \boldsymbol{\varepsilon}_{t+\Delta t}} \boldsymbol{\nabla}^s \boldsymbol{\eta} \, d\Omega \tag{38}$$

If the Dirichlet form (38) verifies (5, 6) then it is regular and is applicable the error estimation methodology described in previous sections.

The local error estimator for this kind of problems is:

$$(E^e_{\Delta t})^2 = \|\boldsymbol{u}^e_{\text{enh}_{t+\Delta t}} - \boldsymbol{u}^e_{h_{t+\Delta t}}\|_E \tag{39}$$

The error bound proposed in (39) is an incremental bound. In order to evaluate the discretisation error along the load path, it is necessary to determine the integral of $E^e_{\Delta t}$ over the time:

$$E^e_{t+\Delta t} = \int_0^{t+\Delta t} E^e_{\Delta t} dt \tag{40}$$

Using the incremental function $W_{t+\Delta t}$, the error estimator is interpreted as the contribution of the incompatible modes of the free energy function:

$$(E^e_{\Delta t})^2 = \int_{\Omega_e} W_{t+\Delta t}\left(\boldsymbol{\varepsilon}^e_{t+\Delta t} - \boldsymbol{\varepsilon}^e_{t+\Delta t}(\boldsymbol{u}), \xi_{t+\Delta t} - \xi_{t+\Delta t}(\boldsymbol{u}), \boldsymbol{\varepsilon}^e_t - \boldsymbol{\varepsilon}^e_t(\boldsymbol{u}), \xi_t - \xi_t(\boldsymbol{u})\right) d\Omega \tag{41}$$

The energy density in (41) can be decomposed in an additive way with the contributions of the elastic and plastic part of the of the free energy, resulting in:

$$\begin{aligned}(E^e_{\Delta t})^2 &= \int_{\Omega_e} W^e_{t+\Delta t}\left(\boldsymbol{\varepsilon}^e_{t+\Delta t} - \boldsymbol{\varepsilon}^e_{t+\Delta t}(\boldsymbol{u}), \boldsymbol{\varepsilon}^e_t - \boldsymbol{\varepsilon}^e_t(\boldsymbol{u})\right) d\Omega + \\ &\int_{\Omega_e} W^p_{t+\Delta t}\left(\xi_{t+\Delta t} - \xi_{t+\Delta t}(\boldsymbol{u}), \xi_t - \xi_t(\boldsymbol{u})\right) d\Omega\end{aligned} \tag{42}$$

The global discretisation error is obtained extending the integral in (41) to the complete domain Ω. Then, the global error is computed via the summation of the local errors:

$$E^2_{\Delta t} = \sum_{i=1}^{n_{el}} (E^i_{\Delta t})^2 \tag{43}$$

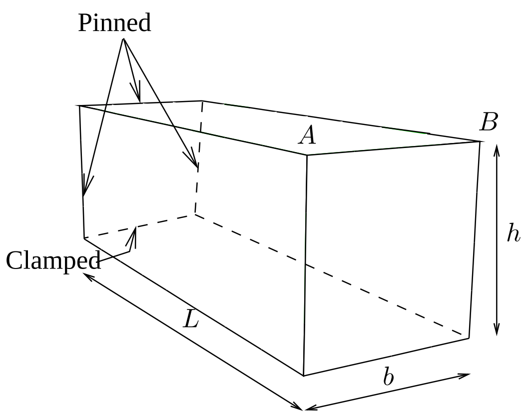
6 NUMERICAL SIMULATIONS

6.1 3-D Finite elasticity. Cantilever beam.

This example analyses the 3D cantilever beam of figure 1, with dimensions $L = 3$, $h = 1$ and $b = 1$. The edge AB has an imposed displacement equal to the depth of the beam h, leading to the deformed mesh showed in figure 1. The hyperelastic constitutive model has the following energy function:

$$W(\boldsymbol{C}) = \frac{1}{2}\lambda(\log J)^2 - \mu\log(J) + \frac{1}{2}\mu(\text{trace}(\boldsymbol{C}) - 3) \tag{44}$$

with $\boldsymbol{C}$ the right Cauchy tensor, J the determinant of the deformation gradient and (λ, μ) the Lamé parameters. The numerical values adopted are: $\lambda = 11538.5$, $\mu = 7692.3$



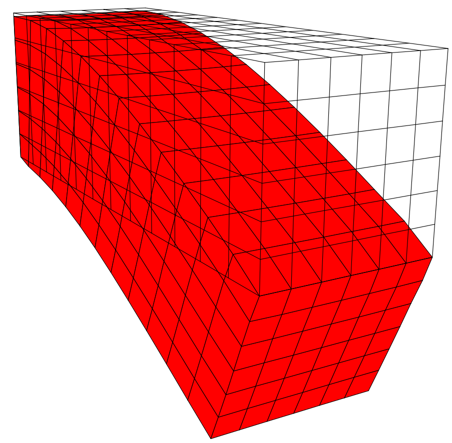


Figure 1: *3D cantilever beam. Geometry, boundary conditions and deformed mesh.*

For error estimation five meshes have been considered with the following elements along length, height and thickness respectively: $2 \times 2 \times 1$, $4 \times 2 \times 2$, $8 \times 4 \times 4$, $12 \times 6 \times 6$ and $16 \times 8 \times 8$.

Figure 2 shows the curves of the energy norm obtained with enhanced elements and the global error estimated at the end of the computation, versus the degrees of freedom considered. The values of the error estimator obtained predict an order of convergence similar to $1/2$: the exact one-half slope plotted in double logarithmic scale is well adjusted to the rate of convergence obtained in the computations.

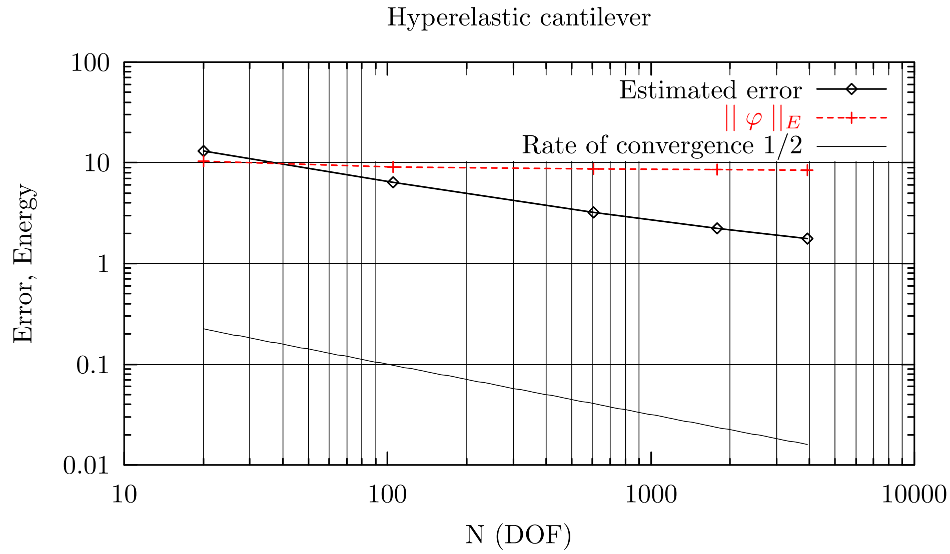


Figure 2: *3D cantilever beam. Evolution of global error and energy norm versus the number of D.O.F.*

Finally, figure 3 shows the local error contours at the end of the process for some of the meshes. The greatest values appears near the edges with imposed displacements (AB and the clamped edge).

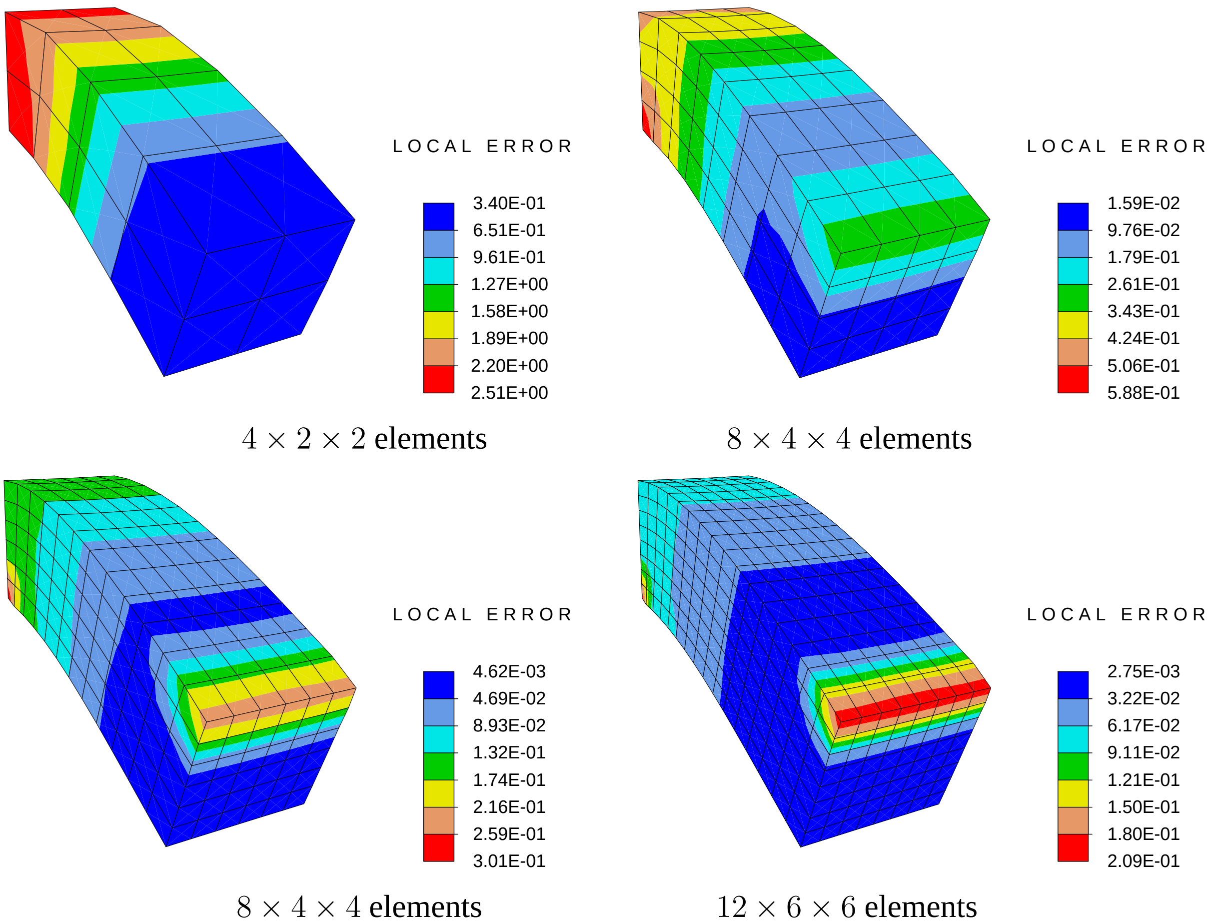


Figure 3: *3D cantilever beam. Contours of local error.*

6.2 Plasticity. Undrained embankment.

The last example concerns a slope stability problem in plain strain. One half of the embankment is considered in the analysis as shown in figure 4, where the vertical face is taken to be a symmetry axis and the lateral surface subtends a $45°$ slope. The embankment, with an increasing gravity load, rests on a rigid surface with no relative displacements over the foundation. The analyses were carried out with meshes of 6×6, 12×12, 24×24, 36×36 and 48×48 elements.

The material was assumed to exhibit undrained response resulting in no changes of volumetric strains during deformation. The elastic properties adopted for the analysis are a Young's modulus $E = 2 \cdot 10^8$, a Poisson's ratio $\nu = 0.25$. The material exhibits elastic-plastic behaviour with no friction angle and initial cohesion $c = 2000$. A constant hardening modulus $H = 2 \cdot 10^3$ is considered relating the yield stress with the effective plastic strain.

The computed force-displacement curves of point A (see figure 4) for each mesh are shown in figure 5. The reference value of the gravity load is 2000. In all the analyses a limit load is predicted by the calculations.

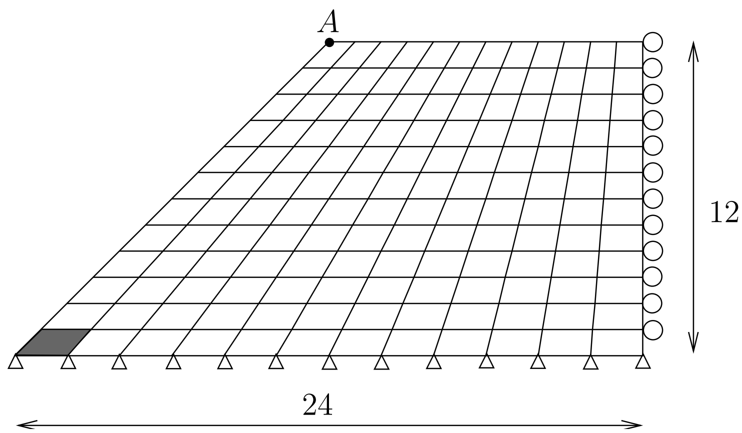


Figure 4: *Undrained embankment. Geometry and boundary conditions.*

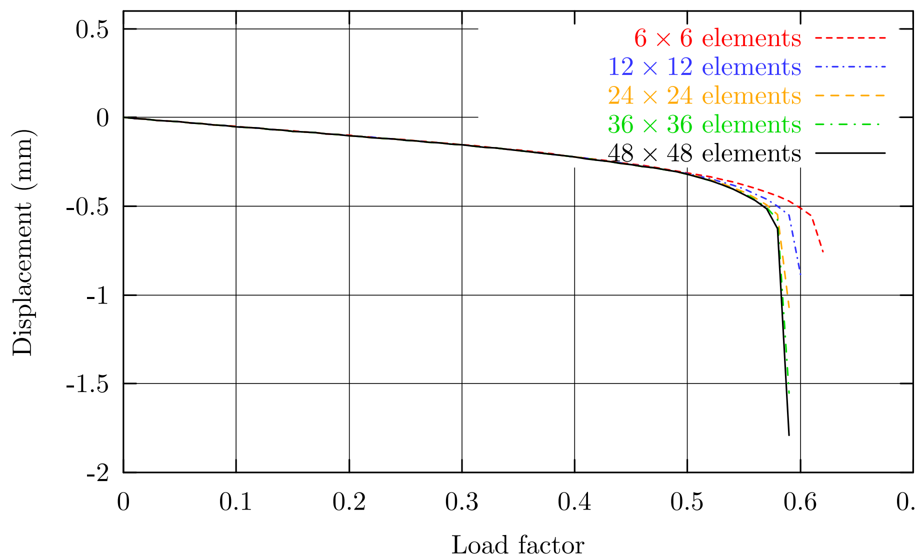


Figure 5: *Undrained embankment. Displacement of upper left corner versus load factor.*

In figure 6 the global error estimator is plotted versus the number of degrees of freedom. The rate of convergence predicted is approximated equal to $1/8$ for the first refinement. It is remarkable that energy increases with refinement whereas the global error decreases as the mesh is refined.

Figures 7 and 8 show the evolution of the elastic part and the plastic part of the accumulated local error computed for the lower left element (shadowed in figure 4). These values are obtained via the additive decomposition of the incremental error expressed in (42). Both of them decrease with refinement of the mesh. Besides, the two components increase during the load process and their order of magnitude are similar. These conclusions are similar to those obtained in other examples ([12],[14])

Finally, figure 9 shows the contours of local error computed for a load factor of 0.53. The value of local error decreases with refinement and tends to localise along a slide line.

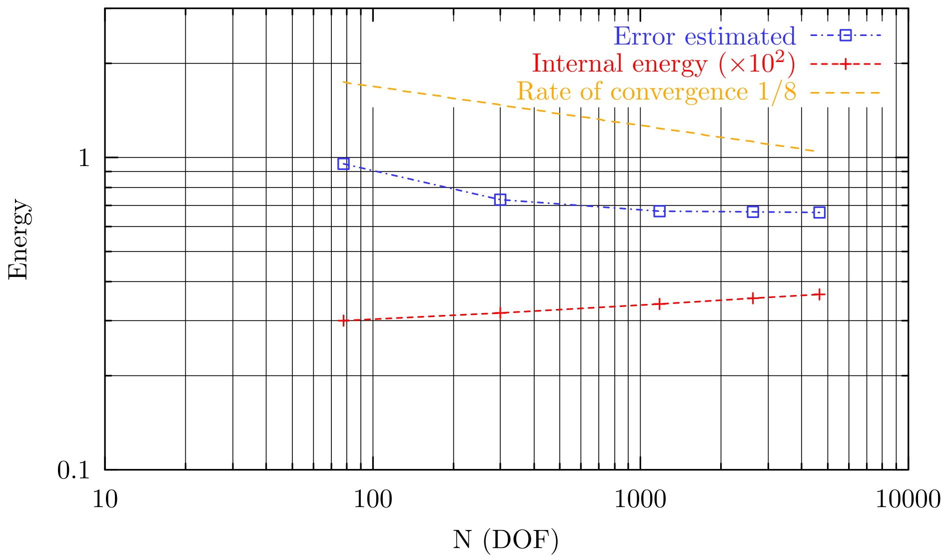


Figure 6: *Undrained embankment. Global error versus D.O.F. (load factor*$= 0.53$*).*

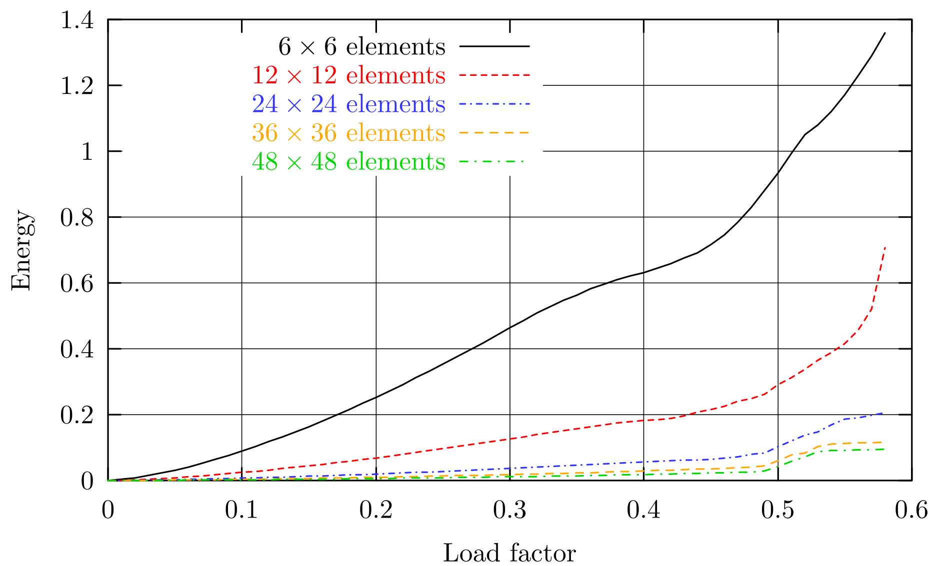


Figure 7: *Undrained embankment. Evolution of the elastic part of accumulated local error for the lower left element.*

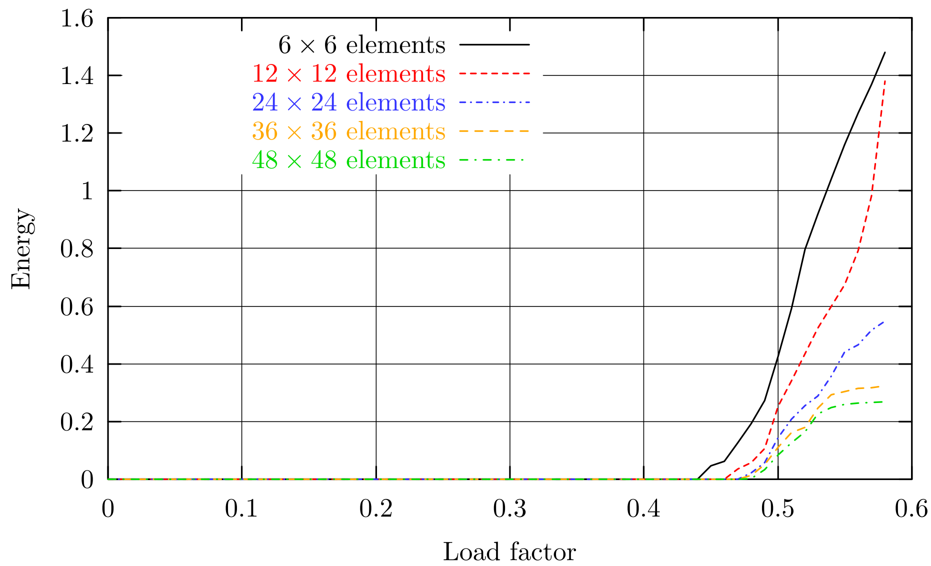


Figure 8: *Undrained embankment. Evolution of the plastic part of accumulated local error for the lower left element.*

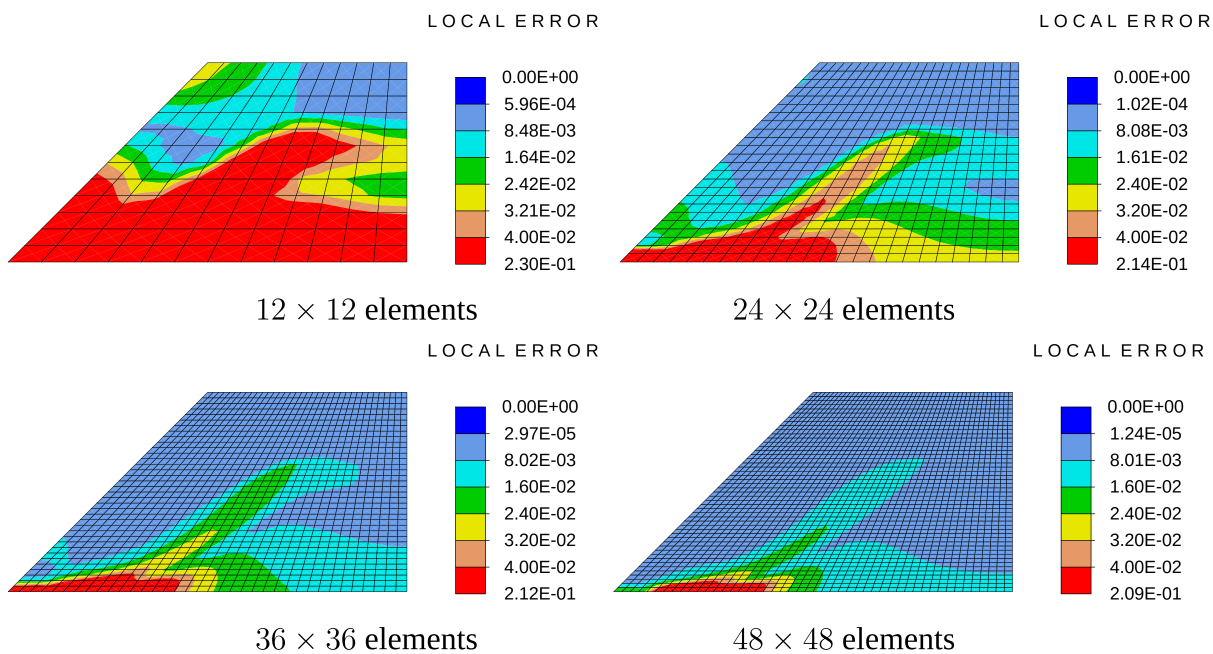


Figure 9: *Undrained embankment. Contours of local error (load factor=0.53).*

7 CONCLUSIONS

A methodology for error estimation valid for linear and non linear problems has been described. The error estimator is based on the energy contribution of incompatible modes and in consequence the estimated error is zero for the patch test strain modes. It has been applied to non-linear finite elasticity and Von-Mises elastic-plastic problems with a formulation which has variational structure at incremental level.

The error estimator proposed establishes a measure of the discretisation error obtained with standard elements, from the solution computed with enhanced assumed strain elements. It is formulated in a local manner and evaluated element by element without smoothing techniques.

Finally, the numerical examples analysed have shown that the results obtained with the proposed method for error estimation are good.

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